$$u = y\sin x - \frac{x^3}{3} + k(y)$$

$$\frac{\partial u}{\partial y} = \sin x + k'(y) = y + \sin x$$

$$\therefore \quad k'(y) = y$$

$$k(y) = \frac{y^2}{2}$$

$$\therefore \quad u(x,y) = c = y\sin x - \frac{x^3}{3} + \frac{y^2}{2}$$

LINEAR EQUATIONS AN INTEGRATING FACTOR.

a general linear d.e. is given by

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \dots + a_0(x)y = 0$$

for linearity all coefficients are functions of x only including the r.h.s. for $n=1$, then it reduces to.

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

if $a_1(x) \neq 0$, then

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

and can be written as

$$\frac{dy}{dx} + P(x)y = R(x) \qquad P(x) = \frac{a_0(x)}{a_1(x)} \qquad R(x) = \frac{g(x)}{a_1(x)}.$$

we need to find a solution to the DE. for given $P(x)$ & $R(x)$.

$P(x)$ & $R(x)$ are continous functions of x and rearranged into the above.

$$[P(x)\,y - R(x)]\,dx + dy = R(x)\,dx$$

multiply both sides by a Familier function of x, $F(x)$

$$F(x)[P(x)\,y - R(x)]\,dx + F(x)\,dy = F(x)R(x)\,dx$$

to find $F(x)$

$$\frac{\partial M}{\partial y} = F(x)P(x) \qquad \frac{\partial N}{\partial x} = F'(x)$$

$$\therefore \quad F'(x) = F(x)\,P(x)$$

seperating variables.

$$\frac{\partial F}{F(x)} = P(x)\,dx$$

$$\ln F(x) = P(x)$$

$$F(x) = e^{\int P(x)\,dx} \quad \Big\}\ \text{integrating factor.}$$

now back to

$$F(x)[P(x)\,y - R(x)]\,dx + F(x)\,dy = F(x)R(x)\,dx$$

$$e^{\int P(x)dx}\left[P(x)\,y - R(x)\right]dx + e^{\int P(x)dx}\,dy = e^{\int P(x)dx}R(x)\,dx$$

$$d\left[e^{\int P(x)dx}\,y\right] = e^{\int P(x)dx}R(x)\,dx$$

EX. $$x \frac{dy}{dx} - 4y = x^6 e^x$$

solve using integrating factor.

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$P(x) = -\frac{4}{x}$ $\quad R(x) = x^5 e^x$ $\quad F(x) = e^{\int -4/x \, dx}$

$$= e^{-4\ln|x|}$$

$$= x^{-4}$$

multiply in integrating factor.

$$x^{-4}\frac{dy}{dx} - 4x^{-5} y = x e^x$$

$$x^{-4} y = \int x e^x \, dx$$

$$x^{-4} y = xe^x - e^x + C$$

$$y = x^5 e^x - x^4 e^x + x^4 C$$

EX. Solve using IF for the IVP. given by:

$\frac{dy}{dx} - 5y = 3e^{5x}$; $y(0) = 8$ $\quad y(1) = ?$

$$F(x) = e^{\int -5 \, dx} = e^{-5x}$$

$$e^{-5x}\frac{dy}{dx} - 5e^{-5x} y = 3e^{5x}(e^{-5x})$$

$$e^{-5x} y = 3x + c$$

$$y = 3xe^{5x} + Ce^{5x}$$

$\therefore C = 8$

$y = 3e^{5x} + 8e^{5x}$

$y(1) = 11e^{5}$

EX.

$\sin y\,dx + (x\sec y - \cos y)\,dy = 0 \quad ; \; y\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$

assume that $x \in (\pi/4, 3\pi/4)$ and $y \in (0, \pi/2)$

going to make this a linear DE of x, not y.

$$\frac{\sin y\,dx}{\sin y\,dy} + \frac{(x\sec y - \cos y)\,dy}{\sin y\,dy} = 0$$

$$\frac{dx}{dy} + \frac{\sec y}{\sin y}x = \frac{\cos y}{\sin y}$$

$$\frac{dx}{dy} + \frac{\sec^2 y}{\tan y}x = \cot y$$

$$F(x) = e^{\int \frac{\sec^2 y}{\tan y}dy} = e^{\ln|\tan y|} = \tan y$$

$$\tan y\frac{dx}{dy} + \sec^2 y\,x = 1$$

$$\tan y\,x = 1 + C$$

$$x = \cot y + \cot y\,C$$

and $\therefore$ $C = \frac{\pi}{4}$

BERNOULLI'S EQUATION.

the DE given by:

$$\frac{dy}{dx} + P(x)y = g(x)\,y^{a}$$

when $a=0$ we can use integrating factor technique.

for $a=1$

$$\frac{dy}{dx} + [P(x) - g(x)]\,y = 0$$

which we can also solve using IF

now we can rewrite Bernoullis equation as.

$$y^{-a}\frac{dy}{dx} + P(x)y^{1-a} = g(x) \quad \text{for } a \neq 0, 1$$

we can write

$$u = y^{1-a} \quad \therefore \quad du = (1-a)\,y^{-a}\frac{dy}{dx}$$

and then Bernoullis becomes.

$$\frac{du}{dx} + (1-a)P(x)\,u = (1-a)\,g(x)$$

and then we can use the IF technique.

EX

Solve the Bernoulli's equation.

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^3}y^{-3}$$

$$\frac{du}{dx} + (4)\frac{1}{x}u = (4)\frac{1}{x^3}$$

$$F(x) = e^{\int 4/x\,dx} = e^{4\ln x} = x^4$$

$$x^4 \cdot u = \int 4x\,dx$$

$$x^4 \cdot u = 2x^2 + C \qquad \therefore u = \frac{2x^2 + C}{x^4} = y^4$$

METHOD OF SUBSTITUTION.

EX.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = e^{-2x}$$

let $\frac{dy}{dx} = u \quad \therefore \quad \frac{du}{dx} = \frac{d^2y}{dx^2}$

$$\therefore \frac{du}{dx} + 2u = e^{-2x}$$

$$F(x) = e^{2x}$$

$$e^{2x}\frac{du}{dx} + 2e^{2x}u = 1$$

$$e^{2x}u = x + C$$

$$\frac{dy}{dx} = xe^{-2x} + C_1e^{-2x}$$

$$y = xe^{-2x} + \frac{1}{2}e^{-2x} - \frac{C_1}{2}e^{-2x} + C_2$$

EX.

$$\frac{dy}{dx} = (x+y)^2 \quad ; \quad y(0) = 0$$

let $u = x + y$

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \therefore \quad \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\therefore \frac{du}{dx} - 1 = u^2$$

$$\frac{du}{dx} = u^2 + 1$$

$$\frac{du}{u^2+1} = dx$$

$$\tan^{-1}(x+y) = x + C$$

$$\therefore C = 0$$

$$\tan^{-1}(x+y) = x$$

$$y = \tan x - x$$

TRIG & HYPERBOLIC

in general

$$re^{i\theta} = r\left[\cos\theta + i\sin\theta\right]$$

$$re^{-i\theta} = r\left[\cos\theta + i\sin\theta\right]$$

and.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

similairly

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

and

$$(e^{i\theta})(e^{-i\theta}) = \cos^2\theta + \sin^2\theta = 1$$

$$(e^{x})(e^{-x}) = 1 = \cosh^2 x - \sinh^2 x.$$

EX. Solve the IVP. given by.

$$\frac{d^2y}{dx^2} = -2y + 2y^3 \; ; \; y(0) = 0 \;\; \& \;\; y'(0) = 1$$

let $v = \frac{dy}{dx}$ $\qquad \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = v\frac{dy}{dx}$

$\therefore \quad v\frac{dv}{dy} = -2y + 2y^3$

$$v\,dv = (-2y + 2y^3)\,dy$$

$$\frac{v^2}{2} = -y^2 + 2y^4 + c$$

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = -y^2 + 2y^4 + C_1$$

$$\therefore \quad C_1 = \frac{1}{2}$$

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = -y^2 + 2y^4 + \frac{1}{2}$$

$$\left(\frac{dy}{dx}\right)^2 = 1 - 2y^2 + y^4 = (1 - y^2)^2$$

$$\frac{dy}{dx} = \pm(1 - y^2)$$

$$\frac{dy}{1 - y^2} = dx$$

$$\tanh^{-1}(1 - y^2) = x + c$$

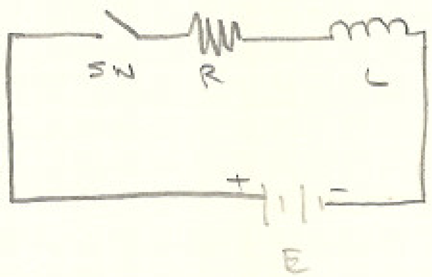
$$y^2 = \tanh(x + c)$$

MODELING

application

electric ciruit

let us consider an R-L series circuit



the switch is closed at $t=0$, assumes that $t \in (-\infty, 0)$ there is no current in the inductor.

ie- $i_L(0)=0$, when sw is closed current will flow in the circuit.

From Len's Law, we have,

$$E = L\frac{di(t)}{dt}$$

From Ohm's Law, we have

$$E = R\,i(t)$$

From Kirchoff's Law, we have

$$E = E_R + E_L$$

$$E = R(i(t)) + L\left(\frac{di(t)}{dt}\right)$$

This is a first order DE.

Now let $R = 20\,\Omega$ $L = 4\,H$ and $E = 100\,V_{DC}$ AT time $t=0$, $i(t)=0$

find the expression for $i(t)$ when $t>0$

$$100 = 20\,i + 4\frac{di}{dt}$$

$$\frac{di}{i-5} = -5\,dt$$

integrate both sides

$$\ln(i-5) = -5t + \ln(c)$$

$$\ln\left(\frac{i-5}{c}\right) = -5t$$

$$\frac{i-5}{c} = e^{-5t}$$

$$i - 5 = ce^{-5t}$$

$$i = 5 + ce^{-5t}$$

using initial conditions, $c = -5$.

$$i = 5 - 5e^{-5t} \quad (A)$$

EX. this time let $E = 20\sin 5t$ volts.

$$20\sin 5t = 20\,i + 4\frac{di}{dt}$$

$$\sin 5t = i + \frac{1}{5}\frac{di}{dt}$$

$$5\sin 5t = 5i + \frac{di}{dt}$$

$$F(x) = e^{\int 5\,dt} = e^{5x}$$

$$5e^{5t}\sin 5t = 5e^{5t}\,i + e^{5t}\frac{di}{dt}$$

$$ie^{5t} = \int 5e^{5t}\sin 5t\,dt$$

let $I = \int 5e^{5t}\sin 5t\,dt$

let $u = e^{5t}$ $\quad dw = 5\sin 5t\,dt$

$du = 5e^{5t}$ $\quad v = -\cancel{\cos}\, 5t\,dt$

$$= e^{5t}\sin 5t + \int 5e^{5t}\cos 5t\,dt$$

$$= e^{5t}\cos 5t + \int 5e^{5t}\cos 5t\,dt$$

let $u = e^{5t}$ $\quad dv = 5\cos 5t\,dt$

$du = 5e^{5t}$ $\quad v = \sin 5t\,dt$

$$= e^{5t}\cos 5t + e^{5t}\sin 5t - \int 5e^{5t}\sin 5t$$

$$= \frac{1}{2}\left[e^{5t}\cos 5t + e^{5t}\sin 5t\right] + c$$

from intial eqations we know $c=0$

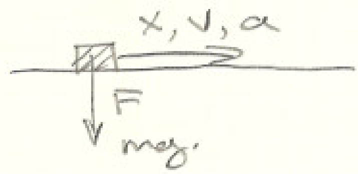
$$i = \frac{1}{2}\left[\cos 5t + \sin 5t\right] + \frac{1}{2}e^{-5t}$$

MASS MOTION ON A HORIZONTAL LINE PLANE

consider a mass sliding on a horizontal line, the force action on the mass is in the direction of the motion.

FROM NEWTON'S SECOND LAW.

$$F = m \cdot \underbrace{\frac{dv}{dt}}_{a}$$



EX. a force of 20N is exerted on a mass of 10 kg which sides on a frictionless planes the mass is initially at rest, calculate the speed of the mass at $t=2$, and the distance ~~d~~travelled.

$$F = m\frac{dv}{dt}$$

$$F\,dt = m\,dv$$

$$Ft = mv + c \qquad \therefore c = 0$$

$$20(2) = (10)v$$

$$\therefore v = 4 \text{ m/s}$$

for distance travelled

$$Ft = mv \qquad v = \frac{dx}{dt}$$

$$Ft = m\frac{dx}{dt}$$

$$Ft\,dt = m\,dx$$

$$\frac{F}{2}t^2 = mx + c \qquad \therefore c = 0$$

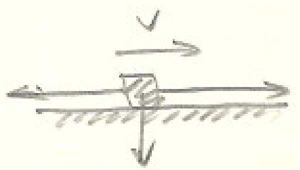
$$\frac{20}{2}(2)^2 = (10)x$$

$$\therefore x = 4\text{ m.}$$

EX. A mass slides on a horizontal plane under the influence of a force F which is given by

$150e^{-2t}\cos 25t$ N

and the friction is given by 6V where v is the velocity of the 3 kg mass, Find the expression for the function of T, then find v (0.5), intinitel velocity has a speed of 20m/s



$$\therefore F - 6v = m\frac{dv}{dt}$$

$$F = m\frac{dv}{dt} + 6v$$

$$150e^{-2t}\cos 25t = 3\frac{dv}{dt} + 6v$$

$$50e^{-2t}\cos 25t = 2v + \frac{dv}{dt}$$

$$F(x) = e^{\int 2dt} = e^{2t}$$

$$50\cos 25t = 2e^{2t}v + e^{2t}\frac{dv}{dt}$$

$$e^{2t}v = \int 50\cos 25t$$

$$ve^{2t} = 2\sin 25t + C$$

$$20e^{0} = 2\sin 25(0) + C$$

$$C = 20$$

$$v = 2e^{-2t}\sin 25t + 20e^{-2t}$$

$$v(0.5) = 2e^{-2(0.5)}\sin 25(0.5) + 20e^{-2(0.5)}$$

$$= \frac{2}{e}\sin(12.5) + \frac{20}{e}$$

$$\cong 7.309 \text{ m/s.}$$

TEMPERTURE RATE OF CHANGE.

under certain conditions, temp. rate of change w.r.t. time is proportional to the difference between body temp (T) and the surrounding temp (T_s), This is the newton law of cooling.

Here it is assumed that T_s stays constant and it is also assumed that T is the same at all points of the body for a given time t. The DE describing this process is:

$$\frac{dT}{dt} = k(T - T_s)$$

constant proportional.

EX. a body whose temp T is intialally at 20°C is imperessed in a liquid @ a temp of 100°C. if the temp of the body is 150° @ t = 1 min what is the temp at 2 min

$$\frac{dT}{dt} = k(T - T_s)$$

$$\frac{dT}{dt} = k(T - 100)$$

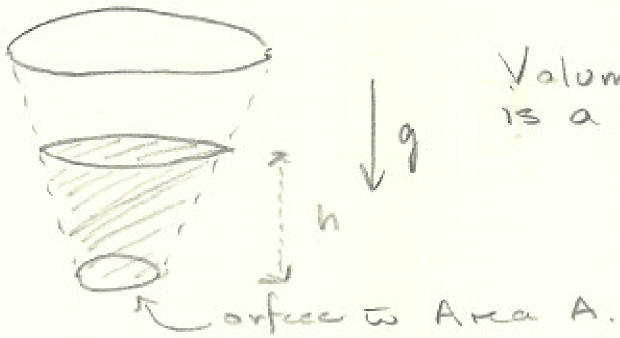
$$\frac{dT}{T - 100} = k\,dt$$

$$\ln|T - 100| = kt + C \qquad \therefore C = \ln(100) = 4.60517$$

$$\therefore k = -0.693147$$

$\therefore T(2\,min) = 125\,°C$

FLOW PROBLEMS



Volume of the vessel/tube is a function of A.

Neglecting frictional loses, then the speed of the escaping water from the orface is

$$\frac{1}{2}mv^2 = mgh$$

m: mass of water
v: speed of flow out.
g: gravity
h: is height of water.

$$v = \sqrt{2gh}$$

$$V = Ah$$

$$dV = A\,dh$$

$$\frac{dV}{dt} = A\frac{dh}{dt} = Av$$

b/c of frictional loses the speed of the water is in fact less then $\sqrt{2gh}$, $h(t)$, an aproximation is

$0.6\sqrt{2gh}$ ft/sec is acceptable

since $V=V(h)$ and $h(t)=h$, then we may have the DE as

$$\frac{dV}{dt} = -\left(0.6\sqrt{2gh}\right)A$$

the negative sign indicates that water is flowing out.

$$\frac{dV}{dt} = -0.6\sqrt{2}\, A\sqrt{g}\, h^{1/2}$$

$$\frac{dV}{dt} = -4.8\, A\, h^{1/2} \qquad \text{where } g = 32.2\ \text{ft/s}^2.$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

and we should be able to solve by seperating variables.

Ex: A tank 4 ft deep, has a rectangular cross section 6ft by 8ft. The tank is initially filled with water which runs out through an orifice of radius of 1" located at the bottom of the tank, find

A. the time required for the tank to empty
B. the time for half the water to empty
C. the height of the water after 20 min.

$$\frac{dV}{dt} = -4.8\, A\, h^{1/2}$$

$$V = (6)(8)h = 48h$$

$$\frac{dV}{dt} = 48\frac{dh}{dt}$$

$$48\frac{dh}{dt} = -4.8\left(\frac{\pi}{12^2}\right)\sqrt{h}$$

$$h^{-1/2}\, dh = -\frac{\pi}{1440}\, dt$$

$$2\sqrt{h} = \frac{-\pi t}{1440} + c$$

$c = 4.$

$$\therefore\ h = \left(2 - \frac{\pi t}{2880}\right)^2 \quad \text{or} \quad t = \frac{2880}{\pi}\left(2 - \sqrt{h}\right)$$

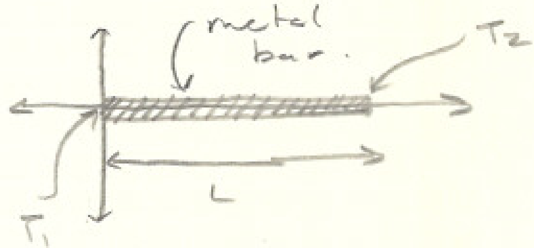
when $h = 0$ all water is drained.

$t = 30.6$ min

when $h = 4/2$, then $t \cong 9.0$ min

when $t = 20$ min, then $h \cong 0.48$ ft.

ONE DIMENSIONAL HEAT FLOW



the metal bar has a cross sectional area of A,
assumptions: one end is at temperature T_1 the other is at T_2

the bar is perfectly insolated latterally so that heat can only flow along the bar.

Find the temperature T, at a distance x between 0 – L. The rate of heat Q at which heat flows across A, is proportional to A and to $\frac{dT}{dx}$ (temp grad) and is given by:

$$Q = -kA\frac{dt}{dx}$$

indicates that heat decreases as x increases.

$T_1 > T_2$

EX.

$T_1 = 60°C$
$T_2 = 0°C$
$L = 100$ cm
$A = 4\ cm^2$

$T(x=0) = 60°$
$T(x=L) = 0°$

$Q = -kA\frac{dT}{dx}$

$dT = \frac{Q}{kA}dx$

$T = \frac{-Q}{kA}x + C$

$\therefore\ C = 60°$

$\therefore\ Q = 0.36$

$T = \frac{-0.36}{(0.15)(4)}x + 60$

$T = -0.6x + 60$

$0 \le x \le 100$

ORTHOGANAL TRAJECTORIES

Ex. show that the curves c_1 & c_2 defined by:

$$y_1 = x_1^3 + c_1 \qquad \& \qquad x_2^2 + 3y_2^2 = 4$$

$$\therefore \frac{dy_1}{dx_1} \cdot \frac{dy_2}{dx_2} = -1$$

$$\frac{dy_1}{dx_1} = 3x_1^2 \qquad \& \qquad 2x_2 + 6y_2\frac{dy_2}{dx_2} = 0$$

$$6y_2\frac{dy_2}{dx_2} = -2x_2$$

$$\frac{dy_2}{dx_2} = -\frac{2x_2}{6y_2}$$

$$3x_1^2 \cdot -\frac{2x_2}{6y_2} = 1$$

$$x_1^2 \cdot \frac{x_2}{y} = 1$$

$$\frac{x^3}{y} = 1$$

pts of intersection are $(-1,-1)$ & $(1,1)$

$\therefore$ they are orthogonal.

ORTHOGANOL TRAGECTORIES

GENERAL METHOD

to find orthoganol tragectories of a given family of curves, we do the following

$$\frac{dy}{dx} = f(x, y)$$ which is the DE that describes the 1st familly of curves

then the DE that describes the second family of curves orthoginal to the first is given by

$$\frac{dy}{dx} = \frac{-1}{f(x,y)}$$

EX.

find the orthogonal tragectories of the family of rectangular hyperbolas given by

$$y = \frac{C_1}{x}$$

solution:

$$\left.\frac{dy}{dx}\right|_{C_1} = \frac{-C_1}{x^2}$$

$$\left.\frac{dy}{dx}\right|_{C_1} \cdot \left.\frac{dy}{dx}\right|_{C_2} = -1$$

$$\left.\frac{dy}{dx}\right|_{C_2} = \frac{x^2}{C_1}$$, but $C_1 = xy$

$$\left.\frac{dy}{dx}\right|_{C_1} = -\frac{y}{x} \qquad \left.\frac{dy}{dx}\right|_{C_2} = \frac{x}{y}$$

then we will solve for the second familly of curves.

$$y\,dy = x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + k$$

$$y^2 - x^2 = C_2 \quad \text{where } C_2 = 2k$$

EX: find the orthogonal trajectories of:

$$r = C_1(1-\sin\theta)$$

SA:

$$\left.\frac{dr}{d\theta}\right|_{C_1} = -C_1\cos\theta \quad , \text{ but } C_1 = \frac{r}{1-\sin\theta}$$

$$\left.\frac{dr}{d\theta}\right|_{C_1} = -\left(\frac{r}{1-\sin\theta}\right)\cos\theta$$

$$\left.\frac{dr}{d\theta}\right|_{C_2} = \frac{1-\sin\theta}{r\cos\theta}$$

$\therefore$

$$r\,dr = \frac{1-\sin\theta}{\cos\theta}\,d\theta$$

$$r = C_2(1+\sin\theta)$$

SOME USEFUL DERIVATIVES

$$d[xy] = x\,dy + y\,dx$$

$$d\left[\ln\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$$

3) $d\left[\frac{1}{2}\ln(x^2+y^2)\right] = \frac{x\,dx + y\,dy}{y^2+x^2}$

4) $d\left[\sqrt{x^2+y^2}\right] = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$

5) $d\left[\sqrt{x^2-y^2}\right] = \frac{x\,dx - y\,dy}{\sqrt{x^2-y^2}}$

6) $d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{x^2+y^2}$

EX.

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$$(y-x)\,dy - (y+x)\,dx = 0$$

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$$

We know that a useful differential is given by

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{x^2+y^2}$$

and

$$d\left[\frac{1}{2}\ln(x^2+y^2)\right] = \frac{x\,dx + y\,dy}{x^2+y^2}$$

Our DE can be writen as.

$$-y\,dx + x\,dx + x\,dy + x\,dy = 0$$

we divid everything by x^2+y^2

$$\frac{x\,dy - y\,dx}{x^2+y^2} + \frac{x\,dx + y\,dy}{x^2+y^2} = 0$$

and we get.

$$\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln(x^2+y^2) = k$$